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EXPLICIT FORMULAS FOR C^n PIECEWISE HERMITE BASIS FUNCTIONS

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INTRODUCTION

We begin with a theoretical characterization of the C^n basis functions with which we are dealing. For a given node x_i in R , there is a basis function $H_{ij}(x)$ associated with the j th derivative of any function f at x_i , where j ranges from 0 to n . In addition, each basis function is nonzero on only two adjacent subintervals. Continuing with the definition of the H 's, we wish an approximation F to f of the form:

$$F(x) = \sum_{j=0}^n H_{ij}(x) f^{(j)}(x_i) + H_{i+1j}(x) f^{(j)}(x_{i+1})$$

where $x_i \leq x \leq x_{i+1}$.

In order that F and its n derivatives will agree with f and n of its derivatives at nodes x_i and x_{i+1} , it is sufficient that the basis functions associated with arbitrary node i obey the following conditions:

$$H_{ij}^{(k)}(x_i) = \delta_{jk}$$

and

$$H_{ij}^{(k)}(x_{i-1}) = H_{ij}^{(k)}(x_{i+1}) = 0$$

where $0 \leq j, k \leq n$.

On a given subinterval each H must obey $n+1$ conditions on the left extreme and $n+1$ conditions on the right extreme. The H 's can therefore be represented by two distinct polynomials of degree $2n+1$.

The following series of pictures depicts the C^4 basis functions (scaled) and their derivatives. The five functions across the top are the basis functions associated with the 0th through the 4th derivatives of f , and the functions underneath them are their successive derivatives. Note that the

functions along the diagonal are nonzero in the center while all off diagonal functions are zero there.

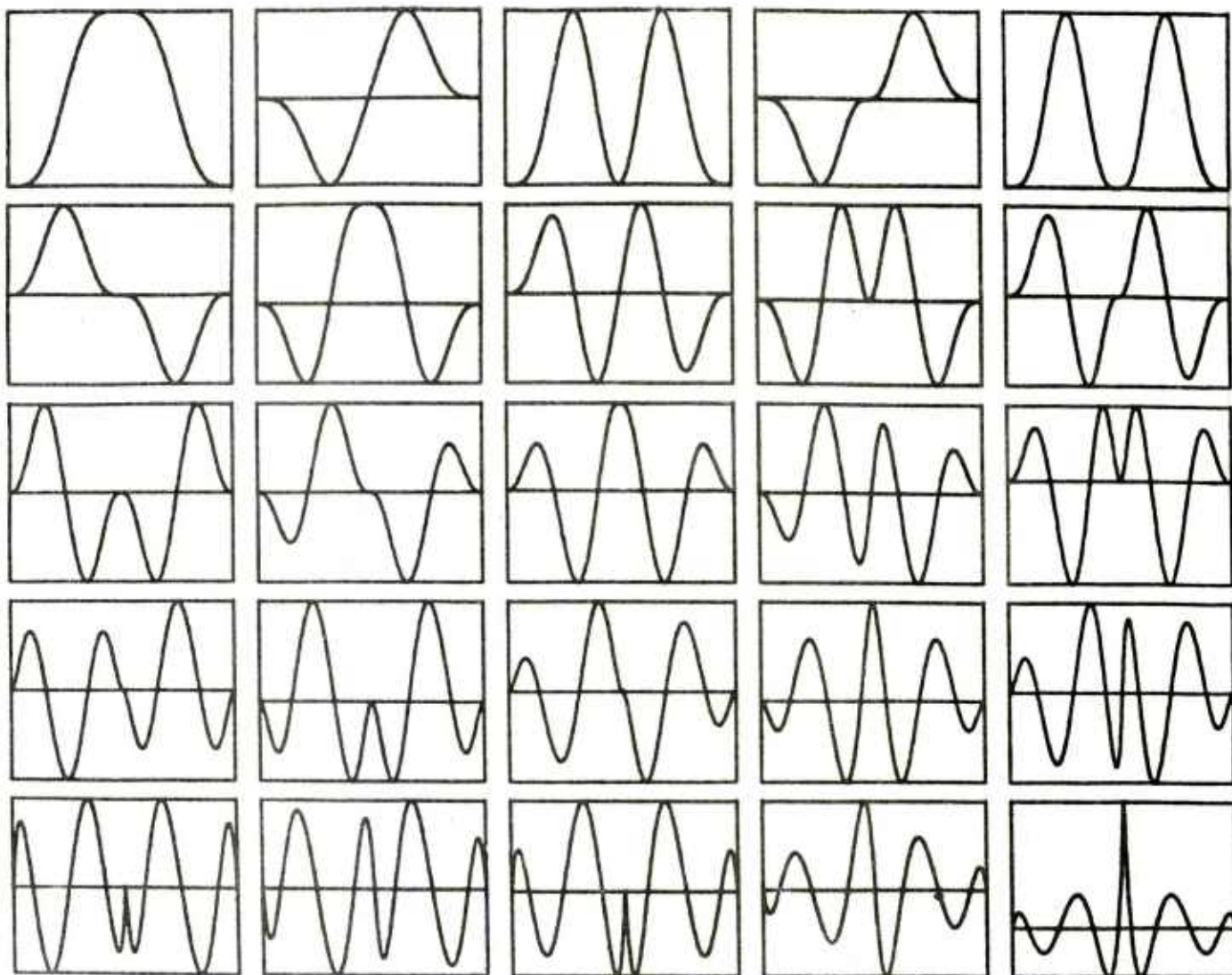


Figure 1. C^4 Basis Functions and Their Derivatives.

Although subsequent analysis will enable us to compute all the basis functions for any given level of smoothness (n) in an extremely simple way, the author has not seen anything similar mentioned or referenced in any finite element text thus far.

DERIVATION

We begin by defining a finite support Taylor series (FSTS). Take the ordinary Taylor series for f around node x_i , truncate it beyond n th derivative terms, and multiply each term by a function which will have the effect of (1) not disturbing the truncated Taylor series at all at node x_i and (2) zeroing the series and n of its derivatives at nodes x_{i-1} and x_{i+1} . If we do this for each node x_i , calling the result $F_i(x)$, we get a global approximation to f which agrees with the truncated Taylor series of f at each node by simply summing the F_i . This is just an alternative way of defining the piecewise Hermite approximation which we will find quite useful.

$$\text{TS: } f(x) = \sum_{j=0}^{\infty} f^{(j)}(x_i)(x-x_i)^j / j! \quad (1)$$

$$\text{FSTS: } F_i(x) = \sum_{j=0}^n f^{(j)}(x_i)(x-x_i)^j g_j(R_i(x)) / j! \quad (2)$$

where

$$\begin{aligned} R_i(x) &= 1 \quad \text{if } x < x_{i-1} \text{ or } x > x_{i+1} \\ &= (x_i - x) / (x_i - x_{i-1}) \quad \text{if } x_{i-1} \leq x < x_i \\ &= (x - x_i) / (x_{i+1} - x_i) \quad \text{if } x_i \leq x \leq x_{i+1} \end{aligned} \quad (3)$$

$R_i(x)$ is just one minus the hat function associated with node i or just the relative position of x in either the left- or the right-hand subinterval. The domain of the g functions is therefore just the interval $[0,1]$.

The objective now is to determine the g 's. Since we want F_1 and its derivatives to behave in a certain manner, we must first differentiate $F_1(x)$ an arbitrary number of times. Using Leibniz's rule for differentiating a product, we have:

$$F_1^{(m)}(x) = \sum_{j=0}^n f^{(j)}(x_1) \sum_{k=0}^{\min\{j,m\}} \binom{m}{k} (x-x_1)^{j-k} g_j^{(m-k)}(R_1(x)) (R_1(x))^{m-k} / (j-k)! \quad (4)$$

and substituting $x = x_1$ in Eq. (4) we have

$$F_1^{(m)}(x_1) = \sum_{j=0}^m f^{(j)}(x_1) \binom{m}{j} g_j^{(m-j)}(0) (R_1(0))^{m-j} \quad (5)$$

We can define $R_1(0)$ to be $R_1(\pm\epsilon)$ or take limits from either side of x_1 .

We now want conditions on the g 's which are sufficient for:

$$F_1^{(m)}(x_1) = f^{(m)}(x_1) \quad (6)$$

and

$$F_1^{(m)}(x_{1-1}) = F_1^{(m)}(x_{1+1}) = 0 \quad (7)$$

for $0 \leq m \leq n$.

We can get these conditions from Eqs. (4) and (5). Conditions on the g 's sufficient for Eqs. (6) and (7) are:

$$g_j(0) = 1 \quad 0 \leq j \leq n \quad (8)$$

$$g_j^{(m)}(0) = 0 \quad 0 \leq j \leq n, 1 \leq m \leq n-j \quad (9)$$

$$g_j^{(m)}(1) = 0 \quad 0 \leq j \leq n, 0 \leq m \leq n \quad (10)$$

The $2n-j+2$ conditions on g_j can be met by a polynomial of degree $2n-j+1$. The product of g_j and $(x-x_1)^j$ in Eq. (2) is therefore of degree $2n+1$ for all j , as expected.

The g of lowest degree is g_n , which has defining conditions:

$$g_n(0) = 1$$

and

$$g_n^{(m)}(1) = 0 \quad 0 < m < n$$

This g , determined by inspection, is:

$$g_n(x) = (1-x)^{n+1} \quad (11)$$

Now, from Eq. (10), we see that all the g 's have the same derivative behavior at $x=1$. We then need only define $g_j(x)$ as the product of some unknown polynomial $h_j(x)$ and $g_n(x)$:

$$g_j(x) = h_j(x)(1-x)^{n+1} \quad 0 < j < n \quad (12)$$

where h_j is a polynomial of degree $n-j$:

$$h_j(x) = \sum_{k=0}^{n-j} a_k x^k \quad 0 < j < n \quad (13)$$

It may seem at first glance that the a 's should have an extra subscript - namely j , since we are seeking $n+1$ sets of coefficients. As will become immediately apparent, however, one set is sufficient and all other sets are subsets of this one. This subset property, along with the fact that the largest set of a 's can be obtained in an almost trivial manner, makes the result of this analysis truly simple indeed.

If we now obtain the m th derivative of $g_j(x)$ and evaluate it at $x = 0$, we get:

$$g_j^{(m)}(0) = m! \sum_{k=0}^m a_k (-1)^{m-k} \binom{n+1}{m-k} \quad (14)$$

Using Eqs. (8) and (9), we have:

$$a_0 = 1$$

and

$$\sum_{k=0}^m a_k (-1)^{m-k} \binom{n+1}{m-k} = 0 \quad \text{for } 1 \leq m \leq n-j \quad (15)$$

This is a lower triangular system, which can be easily solved for the a's by forward substitution. Note that the coefficients do not depend on j; so we might as well solve the largest system (j=0) and obtain all the a's, although only the first n-j+1 a's are needed for h_j(x).

Solving system (15) for j=0 and a few values of n gives us the following table of a's

TABLE I. COEFFICIENTS OF h₀

n	a ₀	a ₁	a ₂	a ₃	a ₄	a ₅
0	1					
1	1	2				
2	1	3	6			
3	1	4	10	20		
4	1	5	15	35	70	
5	1	6	21	56	126	252

Inspection of this table gives us a very simple recursion for the n set of coefficients in terms of the n-1 set:

$$a_0^n = 1$$

$$a_k^n = a_{k-1}^n + a_k^{n-1} \quad 1 \leq k \leq n-1$$

$$a_n^n = 2a_{n-1}^n \quad (16)$$

where the superscripts denote the level of smoothness.

We therefore have here nothing more than one-half of Pascal's triangle, viewed at an angle!

Recalling that:

$$g_j(x) = h_j(x)(1-x)^{n+1}$$

and from the FSTS that:

$$H_{ij}(x) = g_j(R_i(x))(x-x_i)^j/j!$$

we have, explicitly:

$$H_{ij}(x) = h_j(R_i(x))(1-R_i(x))^{n+1} (x-x_i)^j/j! \quad (17)$$

Therefore, f may be approximated on $[x_i, x_{i+1}]$ by:

$$\begin{aligned} & \sum_{j=0}^n \{ f^{(j)}(x_i) h_j \left(\frac{x-x_i}{x_{i+1}-x_i} \right) \left(\frac{x_{i+1}-x}{x_{i+1}-x_i} \right)^{n+1} (x-x_i)^j \\ & + f^{(j)}(x_{i+1}) h_j \left(\frac{x_{i+1}-x}{x_{i+1}-x_i} \right) \left(\frac{x-x_i}{x_{i+1}-x_i} \right)^{n+1} (x-x_{i+1})^j \} / j! \end{aligned} \quad (18)$$

CONCLUSION

In order to evaluate the derivatives of the H 's, one can expand the polynomials involved and multiply out or one can apply Leibniz's rule a couple of times. The latter course is deemed simpler and more numerically stable since it leaves a result which is in "nearly" fully factored form. The latter method was used to produce Figure 1.

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